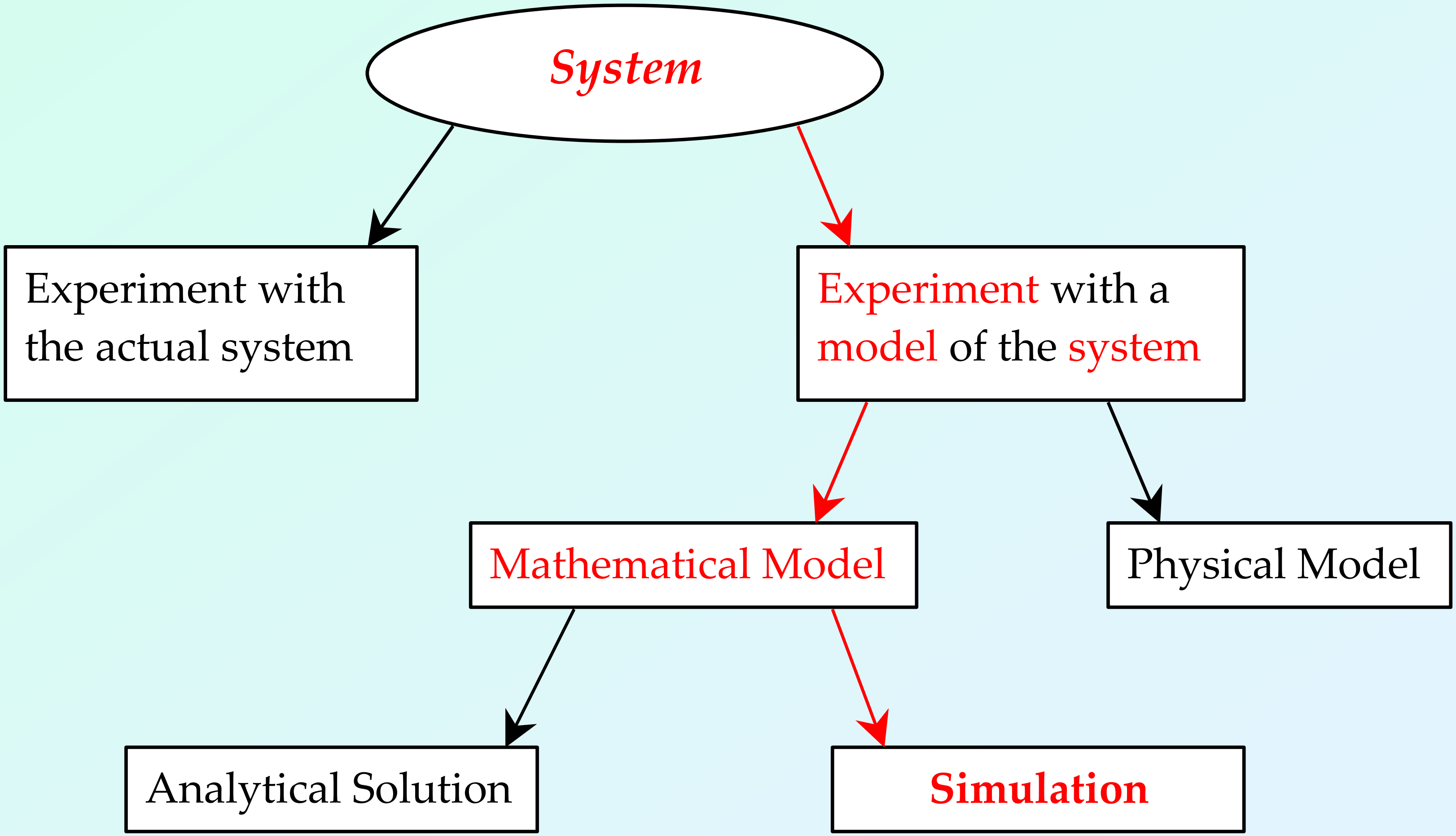
Modeling and Simulation

From observation via implementation to computation and back…

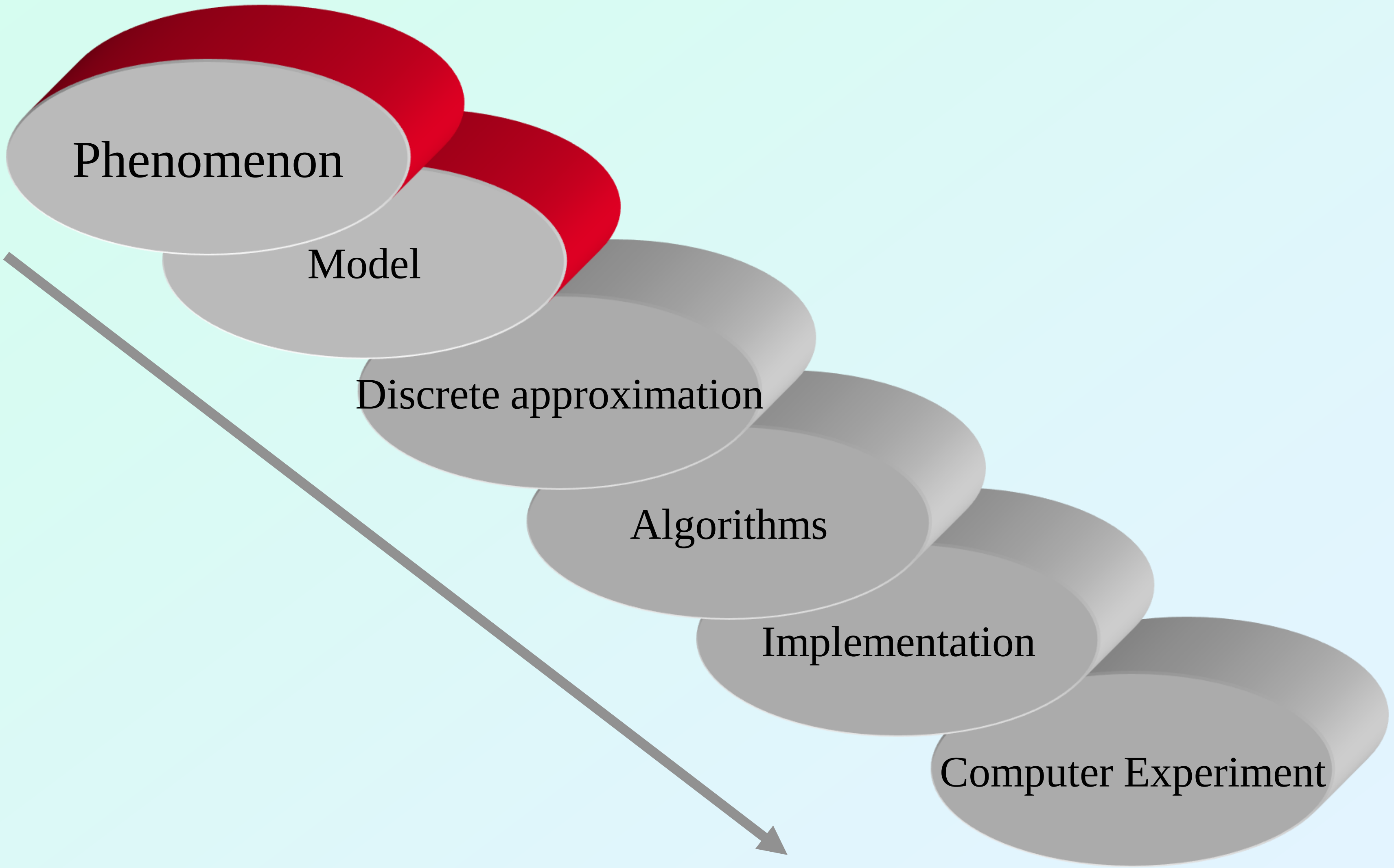
Part Two

Population Dynamics

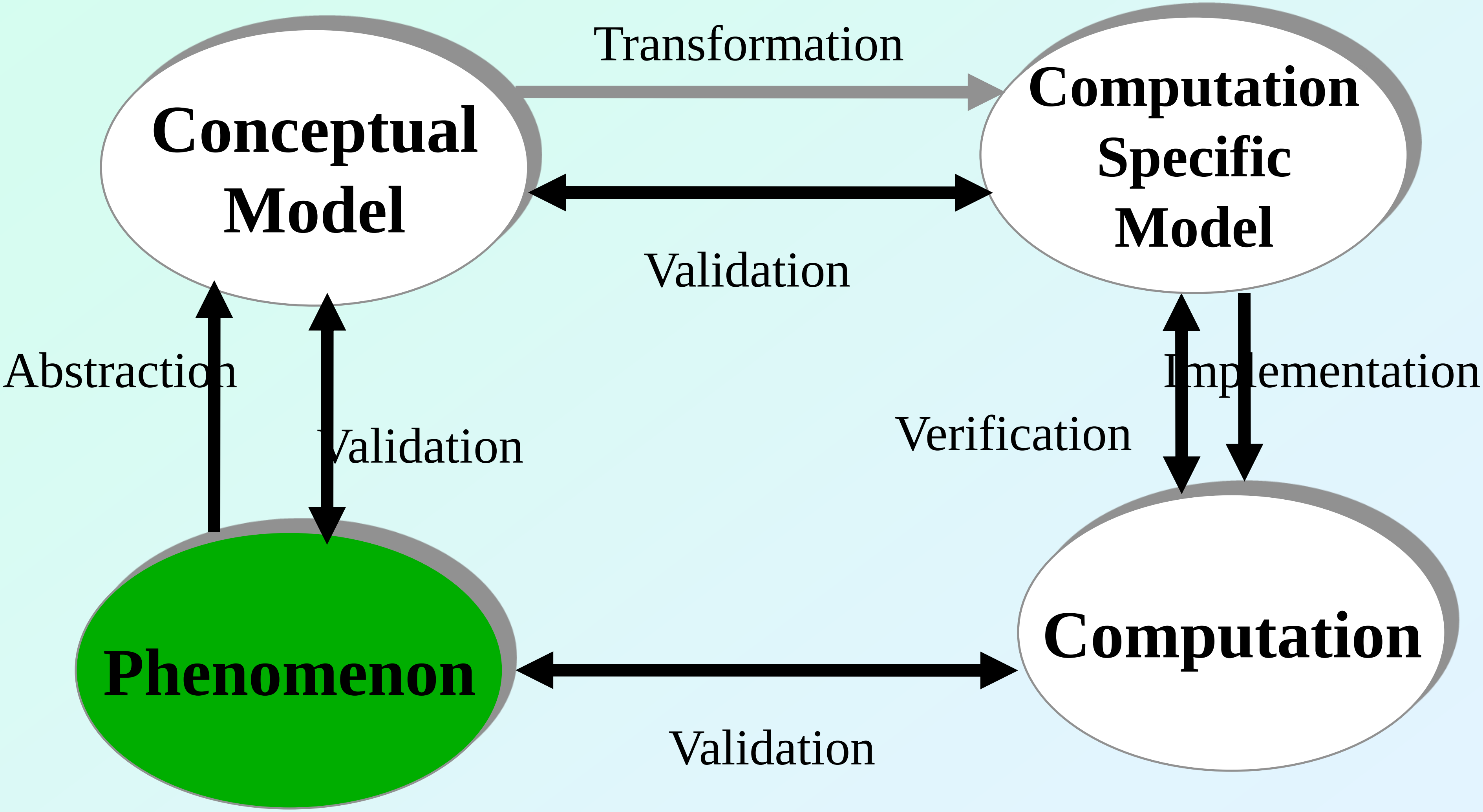
Ways to Study a System



The computer experiment I



From Nature to Computer



Rapid Rabbits

- Fibonacci (Leonardo Pisano; Leonardo di Pisa) 1202
- Rabbits: *Assume*
  - birth in pairs male and female
  - 1 month to maturity
  - 1 month gestational term

Fibonacci

Leonardo van Pisa, beter bekend als **Fibonacci** (ca. 1170 - 1250) was een Italiaanse wiskundige.

Geboren in Italië, opleiding in Noord-Afrika. Hij publiceerde in 1202 "Liber Abaci" (Het boek van de abacus) over algebra en de Arabische cijfers inclusief het cijfer nul. Hij introduceerde dit cijferstelsel hiermee in Europa.

Hij wordt vaak beschouwd als de eerste westerse wiskundige die origineel werk publiceerde sinds de Griekse oudheid. Fibonacci is vooral bekend geworden door zijn rij van Fibonacci.

De naam "Fibonacci" komt van *Figlio di Bonaccio*, "zoon van Bonaccio". *Bonaccio*, "goedzak", was de bijnaam van Fibonacci's vader.

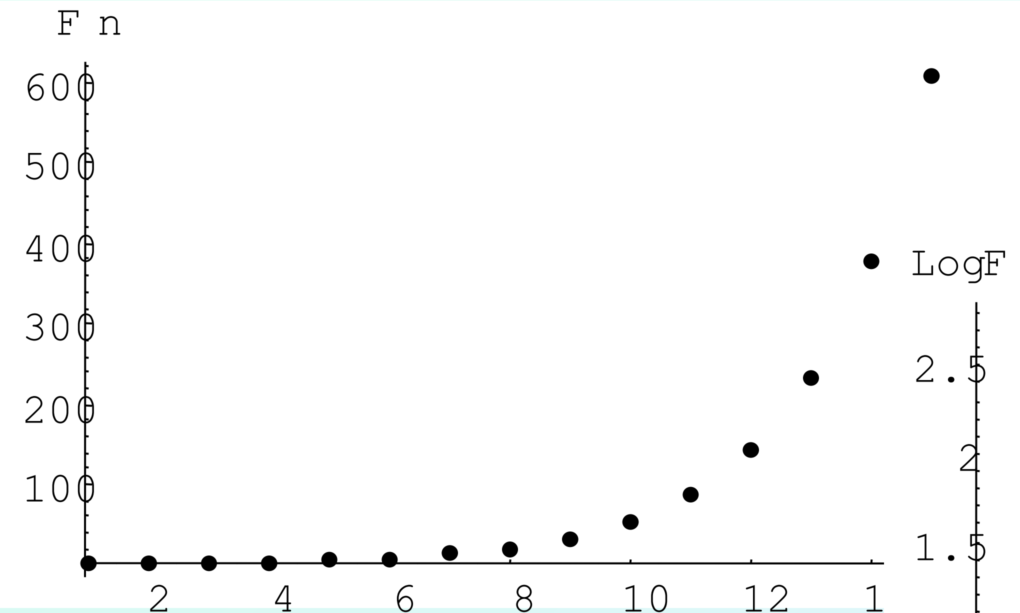
(naar http://nl.wikipedia.org/wiki/Fibonacci)

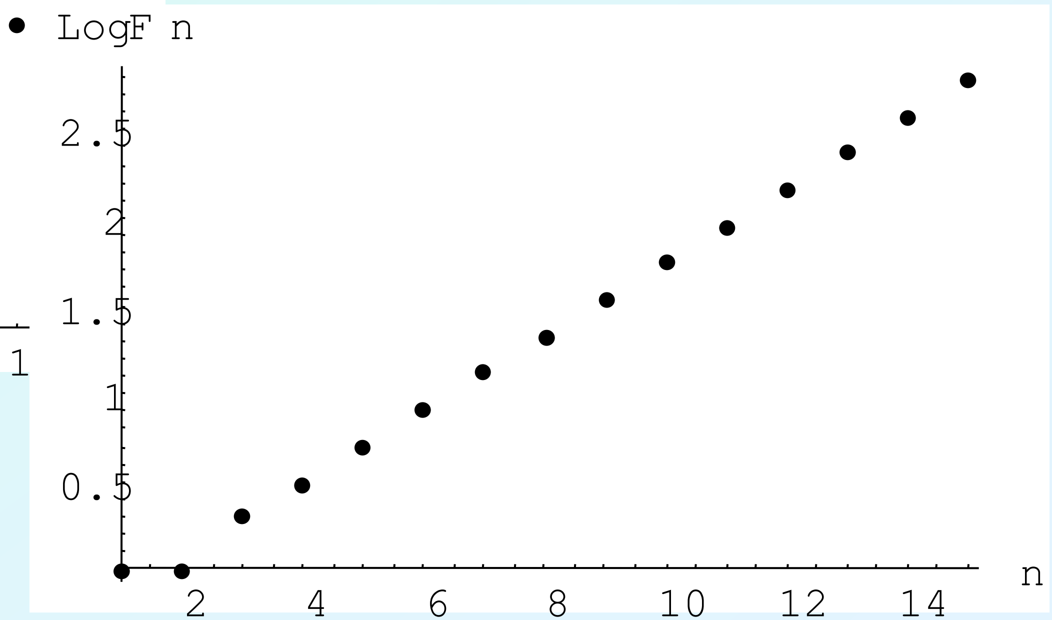
More rabbits...

Pair	Number	Month
[1]	a	1
[1]	a*	2
[2]	a* + aa	3
[3]	a* + aa* + ab	4
[5]	a* + aa* + ab* + aaa + ac	5
[8]	a* + aa* + ab* + aaa* + ac* + ad + aab + aba	6

Fibonacci

F[*n*] = F[*n*-1] + F[*n*-2]





From observation to model

- Question: Growth exponential?
  - Fit
  - Find closed form
- Rewrite:

$$\bar{x}_n = \bar{\bar{A}}\bar{x}_{n-1} \quad \text{With } \bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{and } \bar{\bar{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- If eigenvector unique and distinct: Change to eigenbasis consisting of eigenvectors.

- In the new basis the recurrent relation is decoupled:

$$\overline{\overline{D}} = \overline{\overline{S}}^{-1}\,\overline{\overline{A}}\,\overline{\overline{S}} \text{ with } \overline{\overline{D}} = \begin{pmatrix} 0.5\left(1-\sqrt{5}\right) & 0 \\ 0 & 0.5\left(1+\sqrt{5}\right) \end{pmatrix}$$

$$\overline{\overline{A}} = \overline{\overline{S}}\,\overline{\overline{D}}\,\overline{\overline{S}}^{-1} \text{ and } \overline{\overline{A}}^{n} = \overline{\overline{S}}\,\overline{\overline{D}}^{n}\,\overline{\overline{S}}^{-1}$$

=> the nth Fibonacci term is a linear combination of nth powers of two eigenvalues.

=> $$F(n) \propto c_1 \lambda_1^n + c_2 \lambda_2^n$$

Since $\left|\lambda_1^n\right| < 1$ we have $\mathrm{c}_1 \lambda_1^n \xrightarrow{n \to \infty} 0$

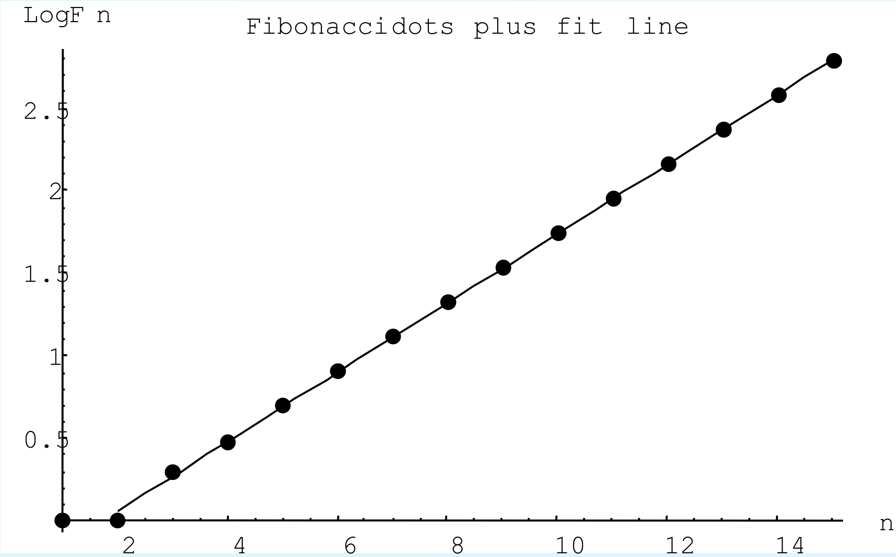
So we expect the rabbit population to grow exponentially!!

$$F(n) \xrightarrow{n\to\infty} c_2 \lambda_2^n$$

Now, fitting our simulations

- Fitting gives

$$\log(F(n)) = -0.366 + 0.211n$$



- The theory tells us
  - for large n.

$$\log(F(n)) = \log(c_2) + \log(\lambda_2)n$$

c_2 = 0.447
λ_2 = 1.618
So

$$\log(F(n)) = -0.349 + 0.209n$$

Explain the small difference between our simulations (fitted results) and the theory

Rabbits don't live forever

- Now, how would Fibonacci's formula change if they only live for n steps?
  - Can you get a stable population?
  - Can you get linear growth?

Malthusian Growth

- Observation: Populations grow exponentially Malthus (1766 - 1834)

$$\frac{dP(t)}{dt} = aP(t) \text{ with a} \sim (birth\_rate - death\_rate)$$

$$\int \frac{dP(t)}{P(t)} = \int a\,dt \Rightarrow P(t) = ce^{at}$$

Thomas Robert Malthus (1766 – 1834)

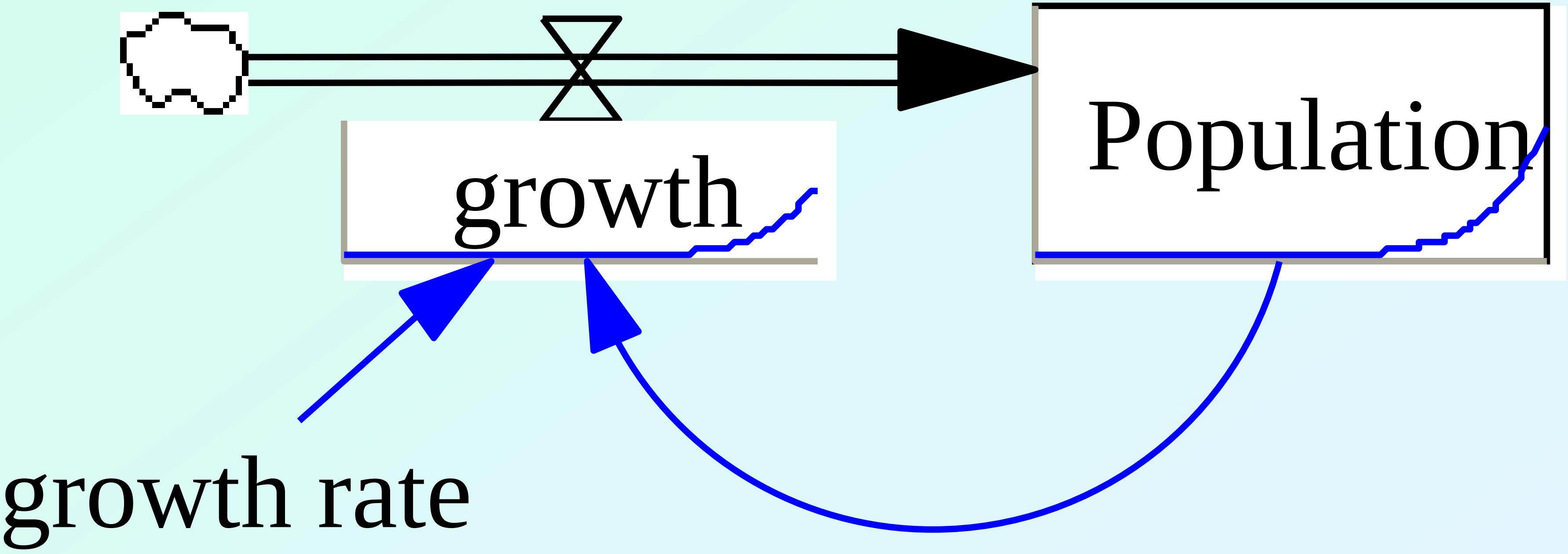
Brits demograaf, econoom en predikant. In 1805 hoogleraar economie in Cambridge. Bekend om zijn pessimistische maar zeer invloedrijke meningen.

Hij wees op de potentiële gevaren van bevolkingsgroei. Als anglicaanse predikant, zag Malthus deze situatie als de manier waarop god de mens deugdzaam gedrag oplegde: hij beschouwde optimistische ideeën van sociale hervormingen als gedoemd te mislukken. De ramp die zich voordoet wanneer de groei van de beschikbare middellen achterblijft bij de bevolkingsgroei staat bekend als een Malthusiaanse catastrofe.

In An Essay on the Principle of Population (1798), stelde hij dat de bevolkingsgroei de economische groei voor zou blijven; hij voorspelde op basis van een eenvoudig model hongersnood op grote schaal. De bevolkingsgroei zou exponentieel zijn, die van de voedselproductie lineair. Hij introduceerde daarvoor de termen Malthusiaans plafond, voor de maximale omvang die de bevolking kan bereiken in verhouding tot de beschikbare grond, en Malthusiaanse catastrofe, waarbij de overbevolking zichzelf in evenwicht brengt door een verhoogde mortaliteit.

(naar http://nl.wikipedia.org/wiki/Malthus)

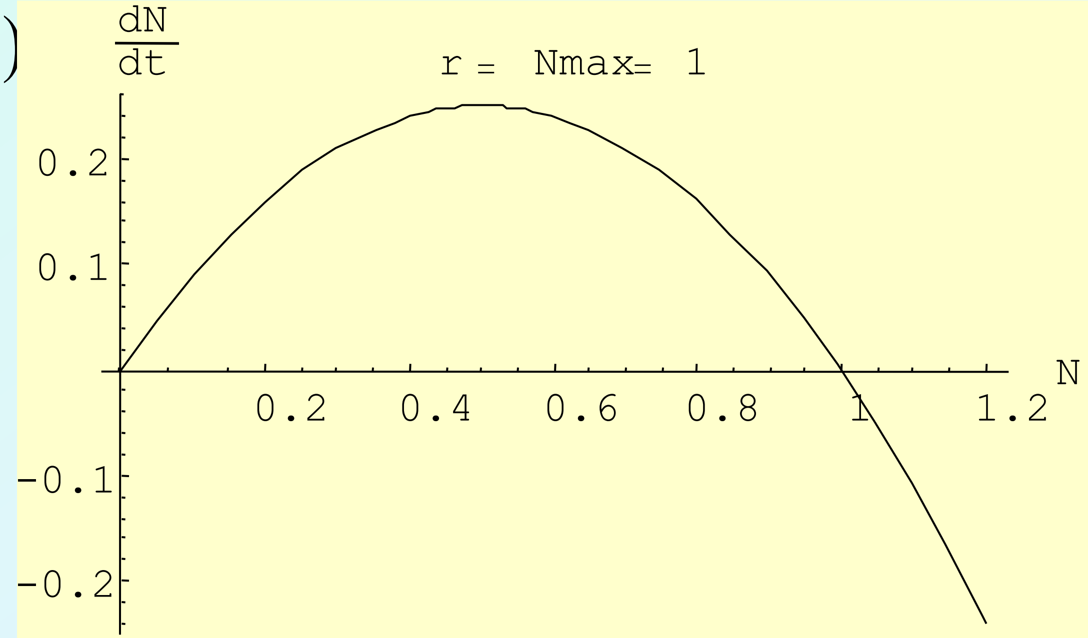
As a Vensim diagram



From Malthus to Verhulst

- Although populations grow exponentially, food resources may be limited! This leads to damping of the growth. E.g.:

$$\frac{dN(t)}{dt} = r\left[1 - \frac{N(t)}{N_{\max}}\right] \times N(t)$$

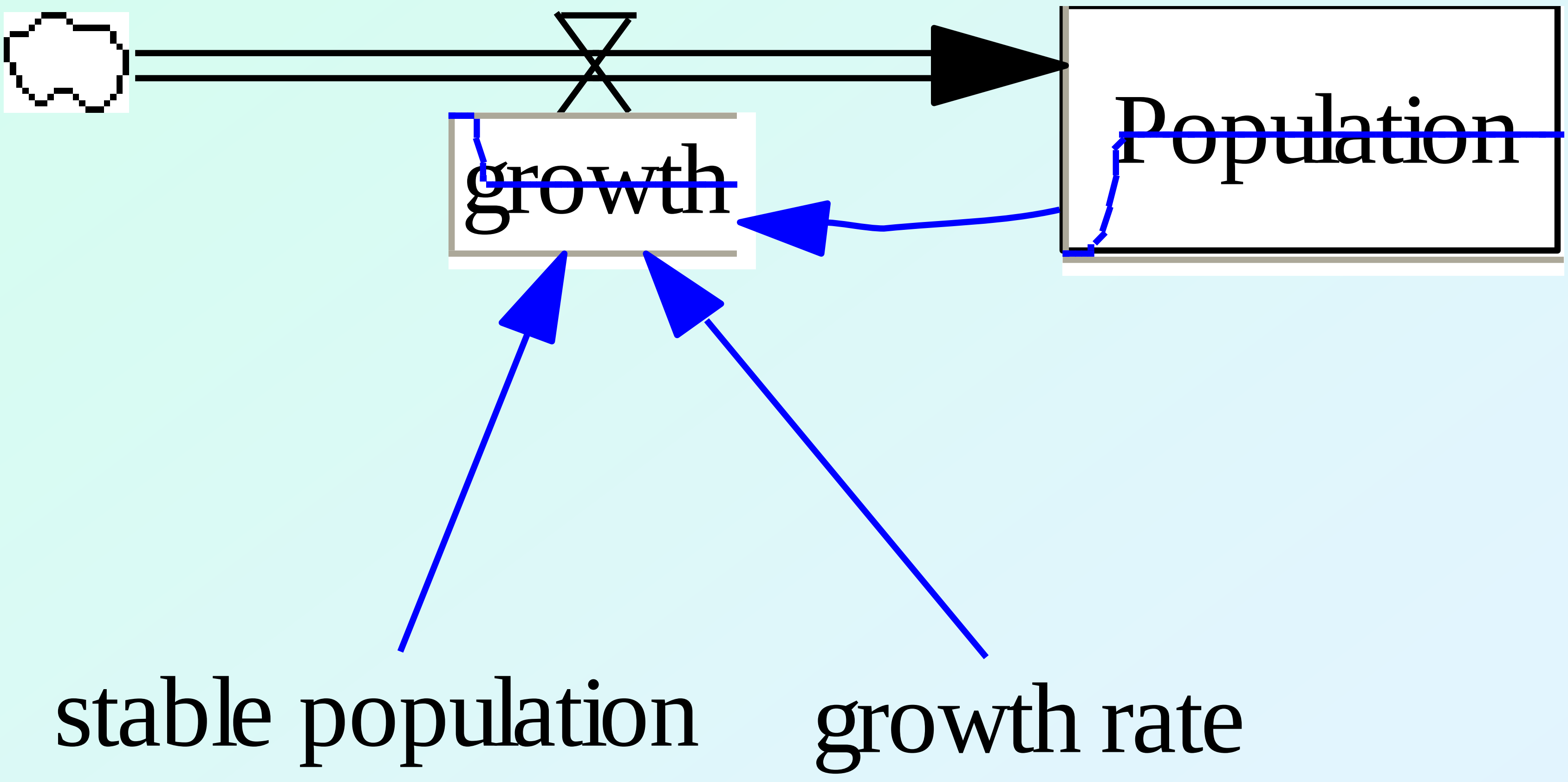


Pierre-François Verhulst

Pierre-François Verhulst (1804 - 1849) Belgisch wiskundige

http://nl.wikipedia.org/wiki/Pierre-Fran%C3%A7ois_Verhulst

Verhulst's model in Vensim

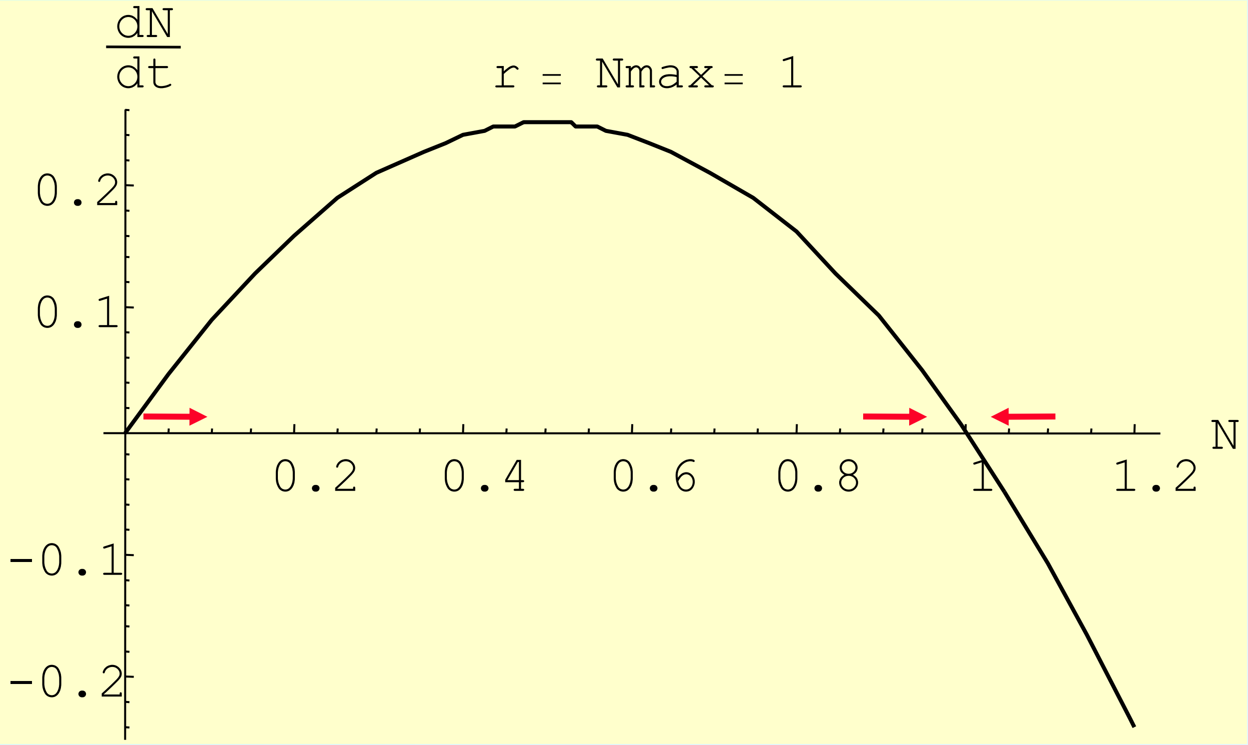


Steady States Logistic Growth

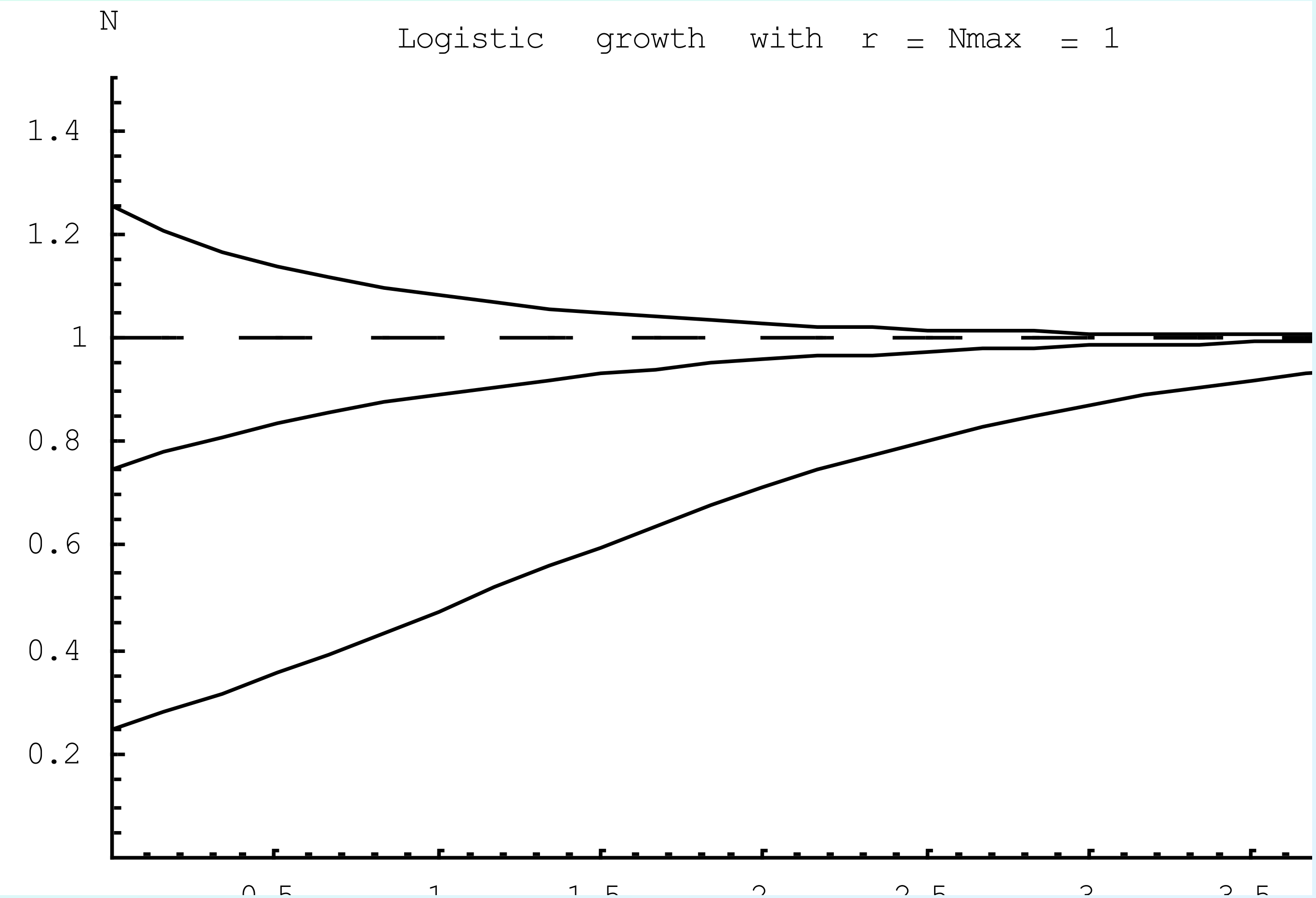
$dN/dt = 0$:

- $N = 0 \rightarrow$ unstable, $dN/dt > 0$ for N > 0.
  - Any small perturbation ε will result in moving away from the steady state.
- $N = N_{max} \rightarrow$ stable,
  - $dN/dt < 0$ for $N > N_{max}$,
  - $dN/dt > 0$ for $N < N_{max}$;
    so, any small perturbation $N_{max} \pm \varepsilon$ will move back to the steady state.

Steady states 2



Solution of ODE for different P(0)



Exact Solution logistic growth

- The logistic growth model has an exact solution, with $N(t=0) = N_0$ it reads

$$N(t) = \frac{N_0 N_{\max} e^{rt}}{N_{\max} + N_0 (e^{rt} - 1)}$$

Can you show that this is a correct solution?
Can you argue that this is the correct solution?

Derivation (simplified)

$$\text{Assume} \frac{dN}{dt} = N(1-N)$$

$$\text{Rewrite as } dt = \frac{dN}{N(1-N)} = \left(\frac{1}{N} + \frac{1}{1-N}\right) dN$$

$$\text{Hence} \int dt = \int \frac{dN}{N} + \int \frac{dN}{1-N} = \int d\log(N) - \int d\log(1-N)$$

$$t = \log(N) - \log(1-N) + c = \log(N/(1-N)) + c$$

$$e^t = aN/(1-N) \;\; ; a = e^c$$

$$N = \frac{e^t}{a(1+e^t)}$$

Reverse (check solution)

$$N(t)=\frac{N_0 N_{\max} e^{rt}}{N_{\max}+N_0(e^{rt}-1)}=\frac{f(t)}{g(t)}$$

$$\frac{dN}{dt}=\frac{f'(t)}{g(t)}-\frac{f(t)g'(t)}{g^2(t)}$$

$$f'(t)=rf(t) \quad ; \quad g'(t)=\frac{rf(t)}{N_{\max}}$$

Hence

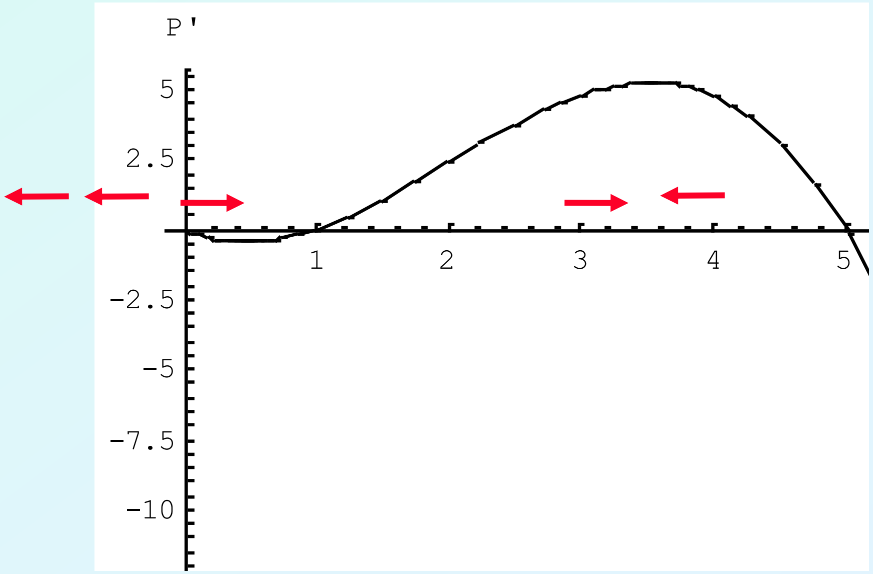
$$\frac{dN}{dt}=\frac{rf(t)}{g(t)}-\frac{rf^2(t)}{N_{\max}g^2(t)}=r\left(N(t)-\frac{N^2(t)}{N_{\max}}\right)$$

$$N(0)=N_0$$

Refining the model

- Observation: 3 stationary points:
  - P=0, T, Pmax, with:
    - 0<P<T : p' <0
    - T<P<Pmax : p' >0
    - P>Pmax: p'<0
    - T is a threshold

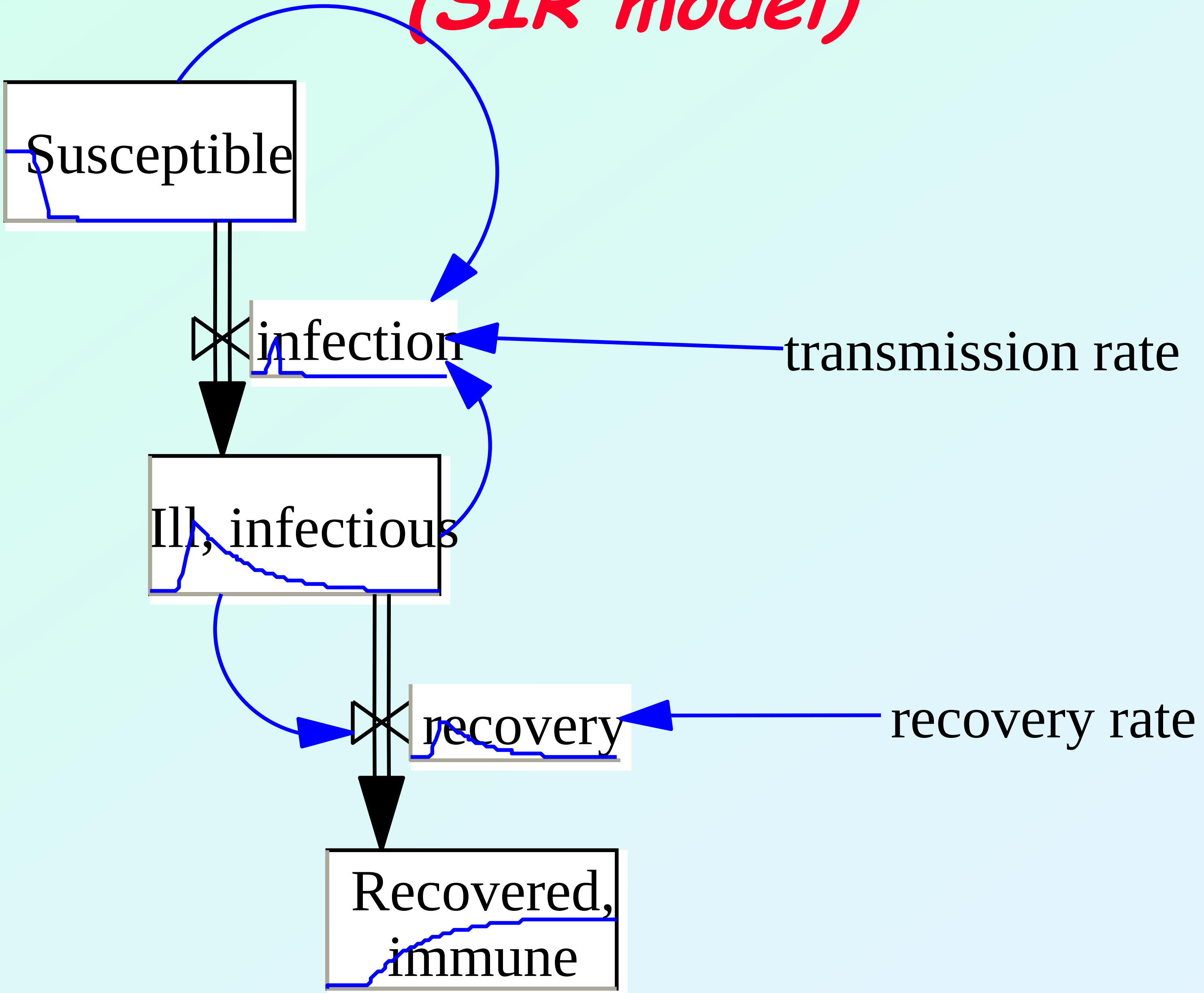
$$\frac{dP}{dt} = k\left[\frac{P}{T} - 1\right]\left[1 - \frac{P}{P_{max}}\right] \times P$$



Interacting populations

Diseases, diseases with vectors,
predator-prey systems,
competing populations

A model for a flu epidemic (SIR model)



Malaria

Malaria is a mosquito-borne infectious disease caused by a eukaryotic protist of the genus *Plasmodium*. It is widespread in tropical and subtropical regions, including parts of the Americas, Asia, and Africa. Each year, there are approximately 350–500 million cases of malaria,[1] killing between one and three million people, the majority of whom are young children in sub-Saharan Africa.[2] Ninety percent of malaria-related deaths occur in sub-Saharan Africa. Malaria is commonly associated with poverty, but is also a cause of poverty[3] and a major hindrance to economic development.

Source: http://en.wikipedia.org/wiki/Malaria

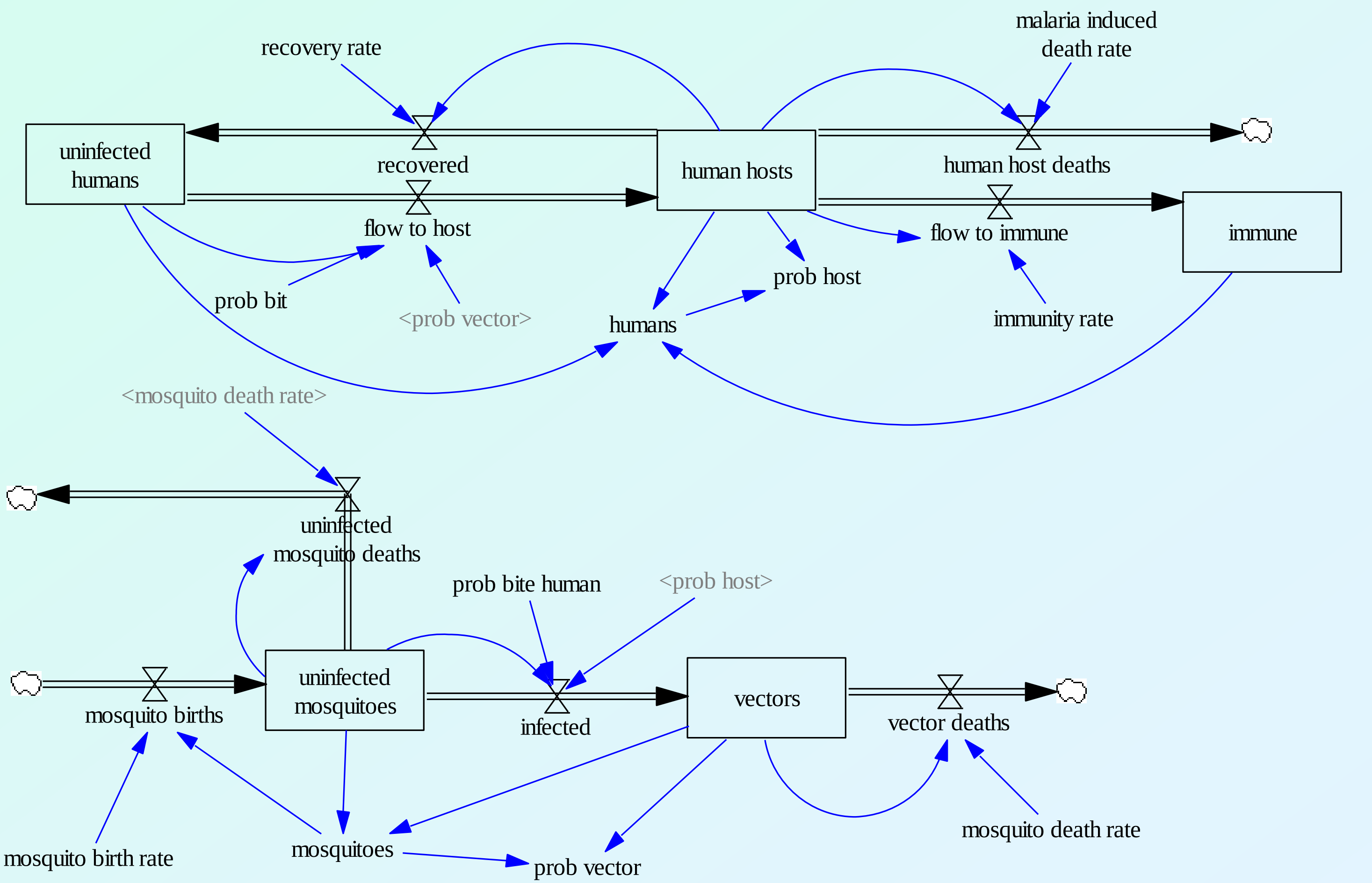
Malaria – what are the relevant populations

- Mosquitoes
  - Uninfected
  - Infected (vector)
- Humans
  - Uninfected
  - Infected/ill
  - Recovered/immune

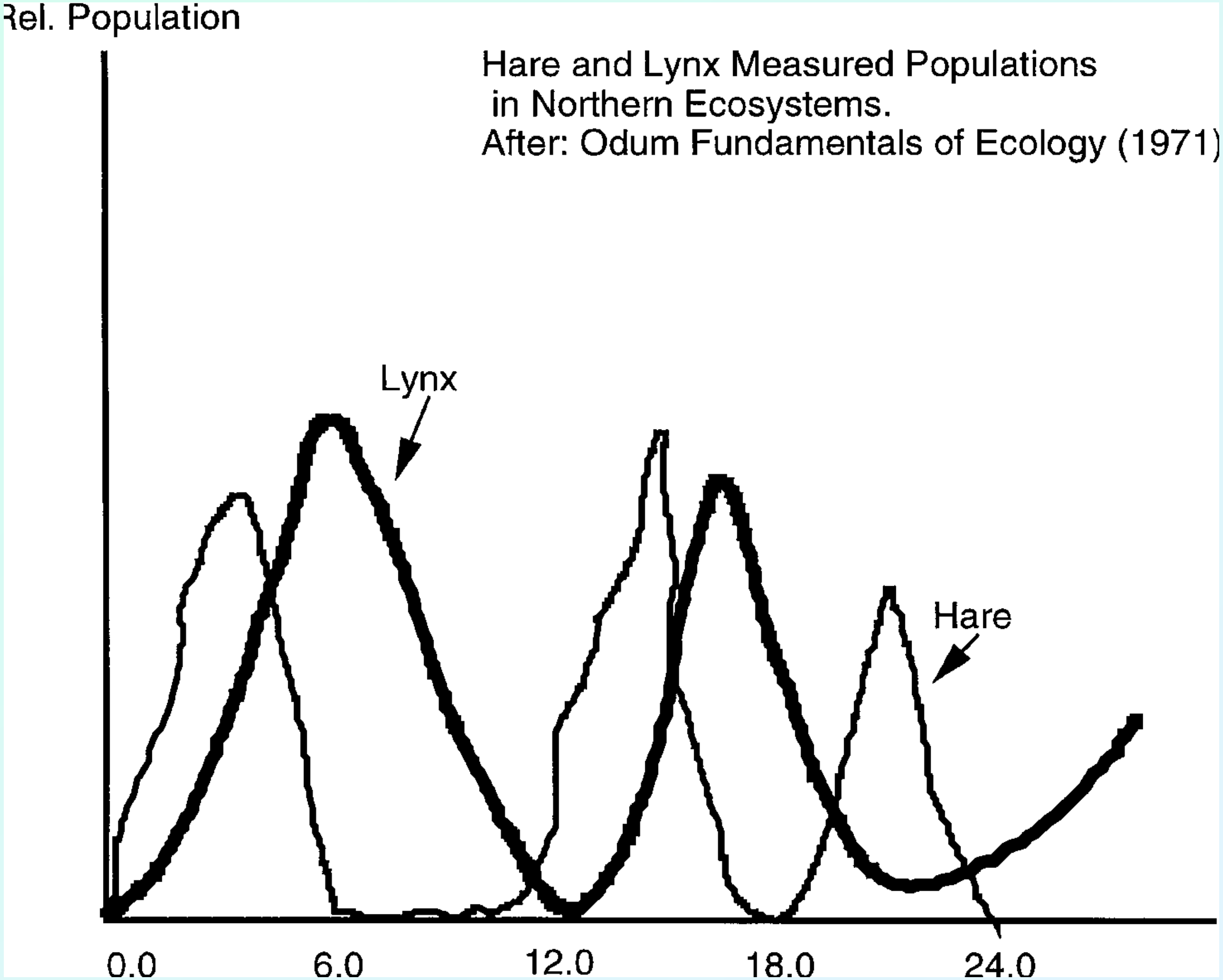
Vensim sketch malaria
(Shiflet & Shiflet)

File: malaria

Closed population of humans. Humans only die from malaria. Mosquitoes go immediately from being uninfected to being vectors.



Two is a crowd: Observation



Conceptual Model

- Describe interactions by product Pred. Prey [x . y]
- Excess deathrate X if no prey [a]
- Excess birthrate Y if no pred [b]
- When predator meets prey energy is exhanged: model efficiency (prey looses more than predator gains) [c]
- Model grazing/foray factor [d]

Mathematical equivalent

$$x' = -ax + cdxy \quad \text{(Predator)}$$
$$y' = by - dxy \quad \text{(Prey)}$$

right turn